A Temperature profile in boundary layers

This appendix is devoted to the calculation of the local temperature of the fluid and its surrounding walls when this system is heated by a constant power source. The heat is generated on the surface of one of the walls and propagates perpendicular to this surface into the fluid and into the heater substrate; i.e. a plane thermal disturbance. The temperature profile is calculated near the heater and near boundaries where heat flows out, both in the fluid and in the walls. The surfaces are considered to be flat thereby allowing the calculations to be restricted to one-dimension perpendicular to these surfaces. It is assumed that, initially, the fluid's temperature is uniform and equal to that of its surrounding walls. The various temperatures are defined as deviations from the uniform initial temperature. The various thermodynamic coefficients are assumed not to vary during heating.

As a first step it is realized that the term in the heat transfer equation (2.21) containing the pressure represents just the contribution of the PE and that this term is not space dependent. Therefore it may be replaced by the time derivative of the uniform bulk temperature $T_b(t)$. In this way, the equations including boundary conditions to be solved for the fluid are:

$$\frac{dT_f(x_i,t)}{dt} - \frac{dT_b(t)}{dt} = D_T\frac{d^2T_f(x_i,t)}{dx^2} \begin{cases} T_f(x_i,0) = 0 & ;x \geq 0 \\ \dfrac{dT_f(0,t)}{dx} = -\dfrac{q_{f,i}(t)}{\lambda} & ;t > 0 \\ T_f(x_i,t) = T_b(t) & ;x \to \infty, t \geq 0 \end{cases}, \quad \text{(A.1)}$$

where λ is the thermal conductivity, D_T the thermal diffusivity and at each wall segment i defined seperately, the space-coordinate x_i and the heat flux per unit surface to the fluid $q_{f,i}(t)$.

For the N different wall segments, we have the usual Fourier equation including boundary conditions for each segment i:

$$\frac{dT_i(x,t)}{dt} = D_i\frac{d^2T_i(x,t)}{dx^2}\begin{cases} T_i(x,0) = 0 & ;x \ge 0 \\ \dfrac{dT_i(0,t)}{dx} = -\dfrac{q_i(t)}{\lambda_i} & ;t > 0 \\ T_i(x,t) \ne \infty & ;x \to \infty, t \ge 0 \end{cases}, \tag{A.2}$$

where $T_i(x,t)$ represents the temperature, $q_i(t)$ the heat flux per unit surface, λ_i the thermal conductivity and D_i the thermal diffusivity of the ith wall segment. The boundary conditions that link the two temperature profiles at each segment i are:

$$T_f(x_i = 0, t) = T_i(0,t) \qquad ;t \ge 0 \tag{A.3}$$

and

$$q_{f,i}(t) + q_i(t) = Q_i(t), \tag{A.4}$$

where $Q_i(t)$ is the amount of heat produced on the surface of the ith wall per unit surface.

In order to solve this set of equations, we introduce the Laplace transforms $\overline{T_f}(x_i, s)$, $\overline{T_i}(x,s)$, $\overline{T_b}(s)$, $\overline{q_{f,i}}(s)$, $\overline{q_i}(s)$ and $\overline{Q_i}(s)$, where

$$\overline{F}(x,s) = \int_0^\infty F(x,t)e^{-st}dt. \tag{A.5}$$

Now, for each wall segment i eqs. (A.1) and (A.2) may be converted into

$$\overline{T_f} = \overline{q_{f,i}}\frac{\sqrt{D_T}}{\lambda}s^{-1/2}\exp\left(-x_i\sqrt{\frac{s}{D_T}}\right) + \overline{T_b} \tag{A.6}$$

and

$$\overline{T_i} = \overline{q_i}\frac{\sqrt{D_i}}{\lambda_i}s^{-1/2}\exp\left(-x\sqrt{\frac{s}{D_i}}\right). \tag{A.7}$$

After some algebra, boundary conditions (A.3) and (A.4) result in:

$$\overline{q_{f,i}} = \overline{Q}F_i - \overline{T_b}s^{1/2}\frac{\lambda_i}{\sqrt{D_i}}F_i, \tag{A.8}$$

where

$$F_i \equiv \left(1 + \frac{\lambda_i}{\lambda}\sqrt{\frac{D_T}{D_i}}\right)^{-1}. \tag{A.9}$$

On approaching the critical point, where λ diverges and D_T vanishes, F_i tends to 1.

First, we proceed with the calculation of the temperature profile at the 'cold walls'. In these walls no heat is produced, so $Q(t) = 0$ and the first term on the r.h.s of eq. (A.8) disappears. Then, substituting eq. (A.8) into eq. (A.6) leads for the ith segment to:

$$\overline{T}_f = \overline{T}_b\left\{1-(1-F_i)\exp\left(-x_i\sqrt{\frac{s}{D_T}}\right)\right\} \tag{A.10}$$

and

$$\overline{T}_i = \overline{T}_b F_i \exp\left(-x\sqrt{\frac{s}{D_i}}\right). \tag{A.11}$$

Equations (A.10) and (A.11) both show that the temperature at the ith wall ($T_f(x_i = 0, t)$ or $T_i(x = 0, t)$) is proportional to the temperature of the bulk ($T_b(t)$), independent of the actual time-profile of $T_b(t)$ and it follows that the ratio between the two is constant in time:

$$\frac{T_f(0, t)}{T_b(t)} = F_i. \tag{A.12}$$

The actual profile near these walls may only be found when the exact behaviour of the bulk temperature is known. In paragraph 2.2.2, the temperature increase of the bulk of the fluid is calculated for a constant heat flux and is given by eq. (2.34). In Laplace space this is:

$$\overline{T}_b = K\sqrt{t_c}\left[s^{-3/2} - s^{-1}\left(s^{1/2} + \frac{1}{\sqrt{t_c}}\right)^{-1}\right], \tag{A.13}$$

where K is a constant proportional to the heat flux.

Substituting eq. (A.13) into (A.10) and (A.11) and transforming to normal space leads to the desired temperature profiles [56]:

$$T_f(x_i, t) = T_b(t) - (1-F_i)Kt_cG(x_i^*, t^*) \tag{A.14}$$

and

$$T_i(x, t) = F_iKt_cG(x^*, t^*), \tag{A.15}$$

with ${}_i^* \equiv x_i/(2\sqrt{D_Tt_c})$, $x^* \equiv x/(2\sqrt{D_it_c})$, $t^* \equiv t/t_c$ and

$$G(p, q) \equiv 2\sqrt{q}\ \mathrm{ierfc}\left(\frac{p}{\sqrt{q}}\right) - \mathrm{erfc}\left(\frac{p}{\sqrt{q}}\right) + \exp(2p+q)\ \mathrm{erfc}\left(\frac{p}{\sqrt{q}} + \sqrt{q}\right). \tag{A.16}$$

Here, ierfc is the integrated complimentary error function erfc:

$$\mathrm{ierfc}(x) = \int_x^{\infty} \mathrm{erfc}(x')dx'. \tag{A.17}$$

For $x = 0$, it follows that $Kt_cG(0, t^*) = T_b(t)$.

Substituting eq. (A.13) into (A.8) with $Q(t) = 0$ and transforming to normal space gives us the heat flow through the ith wall [56]:

$$q_{f,i}(t) = -\frac{\lambda_i}{\sqrt{D_i}}F_iK\sqrt{t_c}\{1 - \exp(t^*)erfc(\sqrt{t^*})\}. \tag{A.18}$$

In order to calculate the temperature profile near the heater, the first term on the r.h.s. of eq. (A.8) must be included. With $i = h$ and $Q(t) = Q_0$ eqs. (A.10) and (A.11) are modified to

$$\overline{T_f} = \overline{T_b}\left\{1-(1-F_h)\exp\left(-x_h\sqrt{\frac{s}{D_T}}\right)\right\} + Q_0F_h\frac{\sqrt{D_T}}{\lambda}s^{-3/2}\exp\left(-x_h\sqrt{\frac{s}{D_T}}\right) \tag{A.19}$$

and

$$\overline{T_h} = \overline{T_b}F_h\exp\left(-x\sqrt{\frac{s}{D_h}}\right) + Q_0(1-F_h)\frac{\sqrt{D_h}}{\lambda_h}s^{-3/2}\exp\left(-x\sqrt{\frac{s}{D_h}}\right). \tag{A.20}$$

When there is no adiabatic temperature rise, i.e. at constant pressure, the usual relations [37] for the heat flow and for the temperature profile are found [56]:

$$q_f(t) = Q_0F_h \tag{A.21}$$

and

$$T_{f,p}(x_h,t) = \frac{Q_0F_h}{\lambda}2\sqrt{D_Tt}\ \mathrm{ierfc}\left(\frac{x_h}{2\sqrt{D_Tt}}\right), \tag{A.22}$$

where the subscript p means at constant pressure. The dissipation ratio q_s/q_f, addressed in section 6.2.2, is equal to $F_h^{-1}-1 = \sigma_h$.

Including adiabatic temperature rise, in order to find the temperature profile in the fluid, we may simply add the two results of eqs. (A.14) and (A.22):

$$T_f(x_i,t) = T_b(t) + \frac{Q_0F_h}{\lambda}2\sqrt{D_Tt}\,ierfc\left(\frac{x_i^*}{\sqrt{t^*}}\right) - (1-F_h)Kt_cG(x_i^*,t^*). \tag{A.23}$$

The same holds for the heat flow and, consequently,

$$q_f(t) = Q_0F_h - \frac{\lambda_h}{\sqrt{D_h}}F_hK\sqrt{t_c}[1-\exp(t^*)\mathrm{erfc}(\sqrt{t^*})]. \tag{A.24}$$

The temperature profile in the heater adds up to

$$T_h(x,t) = F_hKt_cG(x^*,t^*) + \frac{Q_0(F_h-1)}{\lambda_h}2\sqrt{D_ht}\ \mathrm{ierfc}\left(\frac{x^*}{\sqrt{t^*}}\right). \tag{A.25}$$